The diffraction of a rarefaction wave by a corner

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SUMMARY

An investigation is made of the effect of a small disturbance on the flow in a complete rarefaction wave, for example, the flow produced by the rupture of a membrane originally separating a compressible gas from a vacuum. The perturbation arises from a rigid boundary slightly inclined to the direction of flow. The growth of the perturbed region is studied and the pressure field is calculated for diatomic gases.

The nature of the expanding boundary of the perturbed region is investigated. Arguments are put forward which suggest that this boundary can be a weak shock in certain circumstances. A second shock may also appear in some cases, following the first and of greater strength.

In an appendix the solutions are extended to monatomic gases and to fluids with an adiabatic index of 2. The latter results are suitable for a comparison with hydraulic experiments.

1. INTRODUCTION

The term 'complete rarefaction wave' is the name given to the flow produced when a membrane separating a compressible gas from a vacuum is ruptured, or a piston is retracted at the so-called escape speed of the gas (see Courant & Friedrichs 1948, p. 105). If u_1 is the particle velocity and c_1 the speed of sound in the flow, then

$$u_1 = (1-\mu^2)\{(X/t)-c_0\}, \tag{1}$$

$$c_1 = \mu^2(X/t)+(1-\mu^2)c_0, \tag{2}$$

$$\mu^2 = (\gamma-1)/(\gamma+1), \tag{3}$$

where γ is the adiabatic index and c_0 is the speed of sound in the undisturbed gas which previous to the time $t = 0$, when the membrane is removed, occupied the region $X \geqslant 0$.

At the head of the wave adjoining the undisturbed fluid, $X = c_0 t$, and so by equations (1) and (2) $u_1 = 0$, $c_1 = c_0$. The tail of the wave represents the limit of penetration into the vacuum where $c_1 = 0$, and its position is given by

$$u_1 = -\left(\frac{1-\mu^2}{\mu^2}\right)c_0, \qquad \frac{X}{t} = -\left(\frac{1-\mu^2}{\mu^2}\right)c_0.$$

A small perturbation on the flow in the rarefaction wave is produced by a rigid boundary slightly inclined to the direction of flow. The boundary takes the form of two plane surfaces intersecting in the plane $X = 0$. Their equations are respectively $Y = -(\delta_1+\delta_2)X$, for $X < 0$ and $Y = -\delta_2 X$ for $X \geqslant 0$ where δ_1 and δ_2 are small angles (see figure 1). Thus there is a corner in the wall surface at the initial position of the membrane which is concave to the flow for $\delta_1 > 0$. The ensuing flow is assumed to be isentropic. Viscosity and heat conduction are neglected. Thus the only physical constants defining the problem are p_0 and c_0, the pressure and speed of sound in the undisturbed gas and δ_1 and δ_2, the inclination of the walls.

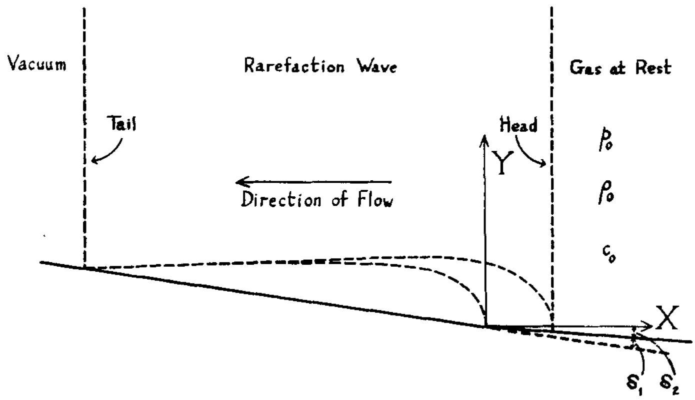

Figure 1. The configuration in the physical plane.

The above conditions have been assumed in order to provide a tractable problem. In fact, rarefaction waves occurring in practice are always incomplete, even under ideal conditions and solutions for such cases are discussed in §7.

2. General solution of the basic equations

It may easily be shown that the velocity potential of the flow in the undisturbed rarefaction wave is

$$\tfrac{1}{2}(1-\mu^2)(c_0^2 t + X^2 t^{-1} - 2c_0 X).$$

If the velocity potential of the flow in the perturbed region is

$$\tfrac{1}{2}(1-\mu^2)(c_0^2 t + X^2 t^{-1} - 2c_0 X) + \phi$$

and squares and products of ϕ and its derivatives are neglected, ϕ satisfies the equation (Chester 1954)

$$\phi_{\eta\eta} - \alpha \frac{x}{\eta} \phi_{x\eta} = \phi_{\zeta\zeta}, \tag{4}$$

where x, η and ζ are new variables given by

$$\left.\begin{aligned} x &= \mu^2(X/c_0 t) + (1-\mu^2) \\ \eta &= \mu^2 X + (1-\mu^2)c_0 t \\ \zeta &= (1-2\mu^2)^{1/2} Y \end{aligned}\right\}, \tag{5}$$

and

$$\alpha = 2\mu^2/(1-2\mu^2). \tag{6}$$

In the problem considered here, ϕ may be further simplified by writing it in the form

$$\phi = c_0 \zeta f(x, y), \tag{7}$$

where

$$y = \eta/(\eta^2 - \zeta^2)^{1/2}. \tag{8}$$

Here the function f is non-dimensional and depends only on non-dimensional combinations of the variables x, η and ζ. This is required because of the lack of a fundamental length or time scale in the data defining the problem. The variable y remains finite and real in the interior of the perturbed region for, as will appear later, $\eta > \zeta$ except at the tail of the rarefaction wave where $\eta = \zeta = 0$. Since $\eta \geqslant 0$ it also follows that $y \geqslant 1$.

Substitution of (7) in equation (4) gives the equation

$$y(y^2-1)f_{yy} - \alpha x(y^2-1)f_{xy} + 3y^2 f_y = 0, \tag{9}$$

with the solution

$$f_y = (y^2-1)^{-3/2}\psi(x^{1/\alpha}y), \tag{10}$$

where $\psi(x^{1/\alpha}y)$ is an arbitrary function of $x^{1/\alpha}y$. From equations (2) and (5), $x = (c_1/c_0) \geqslant 0$ and hence $x^{1/\alpha}$ is real and positive.

Beyond the head of the rarefaction wave, that is for $X \geqslant c_0 t$, $Y \geqslant 0$ or $x \geqslant 1$, $y \geqslant 1$, the gas is at rest. Thus ϕ is identically zero for $x \geqslant 1$, $y \geqslant 1$. The same is true of f_y and hence by (10) $\psi(x^{1/\alpha}y) = 0$ for $x^{1/\alpha}y > 1$. Thus to the present order of approximation no perturbation exists outside the boundary $x^{1/\alpha}y = 1$. Moreover inside this boundary $x \to 0$ as $y \to \infty$ and so, by continuity, it follows from equation (8) that $\eta^2 - \zeta^2$ is of constant sign in the perturbed region. At the wall surface, $\zeta = 0$, so that in the perturbed region $\eta > \zeta$. This is in agreement with the assumption made earlier in this section.

For $x^{1/\alpha}y < 1$, ψ is determined by the boundary condition at the wall surface. In a linearized form, the condition of zero normal velocity becomes

$$\left.\begin{aligned} \frac{\partial\phi}{\partial Y} &= -(\delta_1+\delta_2)(1-\mu^2)\{(X/t)-c_0\} \qquad \text{for } X < 0 \\ \frac{\partial\phi}{\partial Y} &= -\delta_2(1-\mu^2)\{(X/t)-c_0\} \qquad \text{for } X > 0 \end{aligned}\right\}, \tag{11}$$

the velocity along the wall surface being given to the first order by equation (1). Equations (11) are equivalent to

$$\frac{\partial\phi}{\partial\zeta} = -c_0\lambda(x-1)\{\delta_2+\delta_1 H(1-\mu^2-x)\}, \tag{12}$$

where λ is a constant defined by

$$\lambda = \frac{1-\mu^2}{\mu^2(1-2\mu^2)^{1/2}}, \tag{13}$$

and $H(x)$ is the Heaviside unit function defined by

$$\left.\begin{aligned} H(x) &= 0 \qquad x < 0 \\ H(x) &= 1 \qquad x > 0 \end{aligned}\right\}. \tag{14}$$

From equations (7) and (10) it follows that

$$\frac{\partial \phi}{\partial \zeta} = c_0 \int_{x^{-1/\alpha}}^{y} x^{1/\alpha} y \psi'(x^{1/\alpha} y) \frac{dy}{(y^2-1)^{1/2}}, \tag{15}$$

where the prime denotes the derivative of ψ and the lower limit in the integration is chosen so that $\partial\phi/\partial\zeta = 0$ on the boundary $x^{1/\alpha}y = 1$. In their present form equations (12) and (15) combine to give ψ as the solution of a Volterra integral equation of the first kind. It is, however, of a particularly simple form since by the transformations

$$y_1 = \frac{1}{y^2 x^{2/\alpha}}, \tag{16}$$

$$\chi(y_1) = \tfrac{1}{2} y^3 x^{3/\alpha} \psi'(x^{1/\alpha} y), \tag{17}$$

it may be reduced to Abel's integral equation

$$-\lambda(x-1)x^{1/\alpha}\{\delta_2 + \delta_1 H(1-\mu^2-x)\} = \int_1^{x^{-2/\alpha}} \frac{\chi(y_1)}{(x^{-2/\alpha}-y_1)^{1/2}}\, dy_1. \tag{18}$$

This has the explicit solution (Bocher 1914, p. 10)

$$\chi(y_1) = \frac{2\lambda}{\alpha\pi}\frac{d}{dy_1}\int_1^{y_1^{-\alpha/2}} \{\delta_2 + \delta_1 H(1-\mu^2-x)\}\, \frac{(x-1)\, dx}{x^{(1+\alpha)/\alpha}(y_1 - x^{-2/\alpha})^{1/2}}, \tag{19}$$

whence we may obtain

$$\psi(x^{1/\alpha}y) = \frac{2\lambda}{\alpha\pi} x^{1/\alpha} y \int_{xy^\alpha}^{1} \{\delta_2 + \delta_1 H(1-\mu^2-\nu)\}\, \frac{(1-\nu)\, d\nu}{\nu(\nu^{2/\alpha} - x^{2/\alpha}y^2)^{1/2}}, \tag{20}$$

which together with equation (10) gives the complete solution for the region $x^{1/\alpha}y < 1$.

3. The nature of the boundaries when $\gamma = 1{\cdot}4$

It will be convenient henceforth to use the value $\gamma = 1{\cdot}4$ appropriate to a diatomic gas. Equation (20) may then be integrated in terms of elliptic functions, and clearly the results may be similarly developed for more general γ. Those for $\gamma = \frac{5}{3}$ and 2 are given in the Appendix.

When $\gamma = 1{\cdot}4$ we have

$$\mu^2 = \tfrac{1}{6}, \qquad \alpha = \tfrac{1}{2}, \qquad \lambda = 5(\tfrac{3}{2})^{1/2}, \tag{21}$$

and the surface $x^{1/\alpha}y = 1$ which marks the boundary of the perturbed region becomes $xy^{1/2} = 1$. When $\delta_2 = 0$ it is clear from equation (20) that ψ and hence f, is zero for $xy^{1/2} \geqslant \frac{5}{6}$. Thus there is a second surface, $xy^{1/2} = \frac{5}{6}$, inside the first, which marks the limit of the perturbation due to the displacement δ_1.

The equations of the two boundaries in the physical plane are

$$\left(\frac{X}{c_0 t}+5\right)\left\{1-K\left(\frac{X}{c_0 t}+5\right)^4\right\}^{1/2} = \sqrt{24}\,\frac{Y}{c_0 t}, \tag{22}$$

where $K = 6^{-4}$ for the outer boundary, $K = 5^{-4}$ for the inner boundary. They are both characteristics of the original hyperbolic equation (9) and in § 6 arguments are put forward which suggest that in some circumstances these boundaries take the form of weak shocks.

Both boundaries are similar in shape and expand uniformly about the corner. The tail of the rarefaction wave ($X = -5c_0 t$), the wall, and the two boundaries meet in a point which is the furthest penetration downstream of the disturbances. At this point the gradients of the two boundaries are identical and

$$\left(\frac{dY}{dX}\right)_{X=-5c_0 t} = \left(\frac{1}{24}\right)^{1/2}. \tag{23}$$

The upstream extremities of the inner and outer boundaries are respectively the corner $X = Y = 0$, and the intersection of the wall and the head of the rarefaction wave $X = c_0 t$, $Y = 0$. In each case the boundary is perpendicular to the wall. The shape of the curves is given in figure 2.

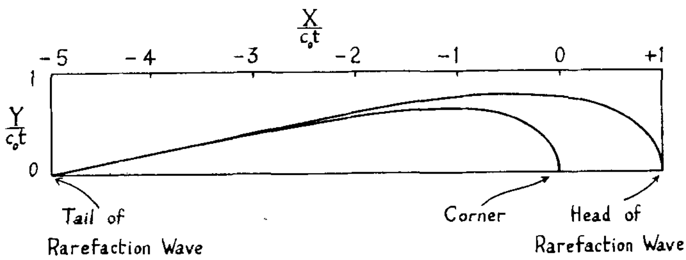

Figure 2. The shape of the two independent regions of perturbed flow.

4. The velocity and pressure fields at the wall surface

For $\gamma = 1{\cdot}4$, equation (20) may be integrated to give

$$f_y = \frac{20}{\pi}\left(\frac{3}{2}\right)^{1/2}\frac{1}{(y^2-1)^{3/2}}\left\{\delta_2[\tfrac{1}{2}\cos^{-1}x^2y - 2^{-1/2}xy^{1/2}\,\mathrm{cn}^{-1}(xy^{1/2})] + \right.$$

$$\left. + \delta_1 H(\tfrac{5}{6} - xy^{1/2})\left[\tfrac{1}{2}\cos^{-1}\left(\frac{36}{25}x^2y\right) - 2^{-1/2}xy^{1/2}\,\mathrm{cn}^{-1}\frac{6xy^{1/2}}{5}\right]\right\}, \tag{24}$$

where cn^{-1} is the inverse function corresponding to the jacobian elliptic function cn of modulus $\frac{1}{2}\surd 2$. It may be expressed alternatively in the form

$$\mathrm{cn}^{-1}x = \surd 2\int_x^1 \frac{dx}{(1-x^4)^{1/2}} = F(\cos^{-1}x), \tag{25}$$

where $F(\cos^{-1}x)$ is the elliptic integral of the first kind of amplitude $\cos^{-1}x$ and modulus $\frac{1}{2}\surd 2$.

The potential throughout the perturbed region is given by an integration of equation (24); however, the pressure and velocity fields at the wall surface may be obtained otherwise. The normal component of the perturbation velocity $\partial\phi/\partial Y$ is given explicitly by equation (11). The tangential component $\partial\phi/\partial X$ is obtained from equations (7) and (8). Thus

$$\frac{\partial\phi}{\partial X} = \frac{(y^2-1)^{1/2}c_0}{6y}\{xf_x - y(y^2-1)f_y\}, \tag{26}$$

or alternatively (since f satisfies equation (9)) in terms of f_y,

$$\frac{\partial\phi}{\partial X} = -\frac{(y^2-1)^{1/2}c_0}{6y}\int_y^{x^{-2}} \{f_y + x[1+\tfrac{1}{2}(1-y^2)]f_{xy}\}\,dy. \tag{27}$$

The integrand is given immediately by equation (24). Furthermore the only contribution to $\partial\phi/\partial X$ at $y = 1$ is due to the value of the integrand in the neighbourhood of $y = 1$. Thus, at $y = 1$,

$$\frac{\partial\phi}{\partial X} = -\frac{10}{3\pi}\left(\frac{3}{2}\right)^{1/2} c_0\Big\{\delta_2[\tfrac{1}{2}\cos^{-1}x^2 - \surd 2x\,\mathrm{cn}^{-1}\,x] + $$
$$+\delta_1 H(\tfrac{5}{6}-x)\left[\tfrac{1}{2}\cos^{-1}\left(\frac{36x^2}{25}\right) - \surd 2x\,\mathrm{cn}^{-1}\,\frac{6x}{5} - \frac{6x^2}{(5^4-6^4x^4)^{1/2}}\right]\Big\}. \tag{28}$$

There is a singularity in the tangential velocity component at the corner where

$$\frac{\partial\phi}{\partial X} \sim \frac{5}{6}c_0\frac{\delta_1}{\pi}\left(-\frac{6}{5}\frac{X}{c_0 t}\right)^{-1/2} \quad \text{as } X \to -0. \tag{29}$$

As $x \to 0$ or $X \to -5c_0 t$, the perturbation velocity components tend to the finite non-zero limits,

$$\frac{\partial\phi}{\partial X} = -\frac{5\surd 6c_0}{12}(\delta_1+\delta_2), \qquad \frac{\partial\phi}{\partial Y} = \frac{5c_0}{6}(\delta_1+\delta_2),$$

giving a first-order velocity discontinuity across the boundary of the perturbed region at the tail of the wave.

To the present order of approximation the flow is isentropic and the pressure field is determined explicitly by the change in sound speed. Denote the pressures in the perturbed and unperturbed rarefaction wave by p_2 and p_1 respectively. Then $p_2 = p_1$ when $\delta_1 = \delta_2 = 0$. Let the speed of sound in the disturbed region of the rarefaction wave be $c_1 + \Delta c$. Then

$$\frac{p_2-p_1}{p_1} = \frac{7\Delta c}{c_1}. \tag{30}$$

The increase in sound speed Δc is obtained from Bernoulli's equation. Chester (1954) has shown that, in terms of the perturbation potential ϕ,

$$\Delta c = -\frac{1}{c_1}\left\{\frac{1}{5}\frac{\partial\phi}{\partial t} + \frac{1}{6}\left(\frac{X}{t} - c_0\right)\frac{\partial\phi}{\partial X}\right\}. \tag{31}$$

By a transformation into the (x,y)-plane this gives

$$\Delta c = \frac{(y^2-1)^{1/2}c_0}{6y}\int_y^{x^{-2}} \{f_y - x[\tfrac{1}{5}-\tfrac{1}{2}(1-y^2)]f_{xy}\}\,dy, \tag{32}$$

with f_y given by equation (24). Consequently the increase in sound speed is determined on $y = 1$ by the behaviour of the integrand at $y = 1$. From equation (30) the pressure distribution at the wall surface is

$$\frac{p_2-p_1}{p_1} = \frac{35\surd 6}{3\pi}\Big\{\delta_2\left[-\frac{4}{5\surd 2}\,\mathrm{cn}^{-1}\,x + \frac{1}{2x}\cos^{-1}x^2\right] +$$
$$+\delta_1 H(\tfrac{5}{6}-x)\left[-\frac{4}{5\surd 2}\mathrm{cn}^{-1}\,\frac{6x}{5} + \frac{1}{2x}\cos^{-1}\left(\frac{36x^2}{25}\right) + \right.$$
$$\left. + \frac{6x}{5(5^4-6^4x^4)^{1/2}}\right]\Big\}. \tag{33}$$

The pressure is singular on the downstream side of the corner where

$$\frac{p_2-p_1}{p_1} \sim \frac{7\delta_1}{5\pi}\left(-\frac{6X}{5c_0 t}\right)^{-1/2} \quad \text{as } X \to -0. \tag{34}$$

Both p_2 and p_1 remain finite and approach zero as $x \to 0$ but the non-dimensional pressure $(p_2-p_1)/p_1$ is singular.

Values of $(p_2-p_1)/(p_1\delta_1)$ and $(p_2-p_1)/(p_1\delta_2)$ are exhibited in figure 3 for the respective cases $\delta_2 = 0$, $\delta_1 = 0$. This serves to separate the respective contributions due to the two displacements δ_1 and δ_2.

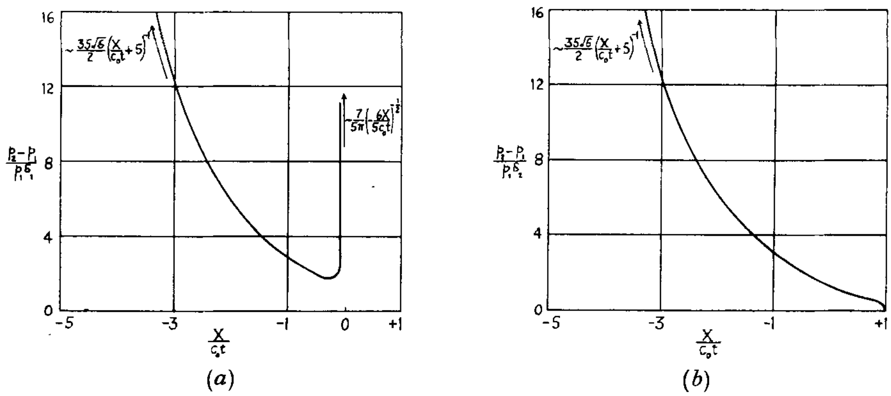

Figure 3. The pressure distribution at the wall surface. (*a*) $(p_2-p_1)/(p_1\delta_1)$ for the case $\delta_2 = 0$. (*b*) $(p_2-p_1)/(p_1\delta_2)$ for the case $\delta_1 = 0$.

5. The sonic line

In the absence of perturbation, the fluid reaches sonic speed on the plane $X = 0$ where $u_1 = -c_1 = -5c_0/6$. The perturbed region due to the displacement δ_1 lies entirely downstream of this plane. Consequently, such a displacement does not alter the position of the sonic line (this is not strictly correct near the corner; in § 6 it is shown that a concave corner causes small disturbances to propagate upstream). Thus we may take $\delta_1 = 0$ with no loss of generality.

In the perturbed region the sonic line is given by the equation

$$(c_1+\Delta c)^2 = \left(u_1+\frac{\partial\phi}{\partial X}\right)^2+\left(\frac{\partial\phi}{\partial Y}\right)^2.$$

By a substitution from equations (1) and (2) for u_1 and c_1, this becomes

$$\frac{X}{t}+\Delta c+\frac{\partial\phi}{\partial X} = 0, \tag{35}$$

correct to the first order, where Δc and $\partial\phi/\partial X$ are to be evaluated on $X = 0$. Equations (27) and (32) give the sonic line as

$$\frac{X}{c_0 t} = \frac{(y^2-1)^{1/2}}{6y}\int_y^{36/25} f_{xy}\, dy. \tag{36}$$

Hence by a substitution from equation (24) for f_y,

$$\frac{X}{c_0 t} = \frac{-5\sqrt{3}}{3\pi}\,\delta_2\,\frac{(y^2-1)^{1/2}}{y}\int_y^{36/25}\frac{\nu^{1/2}}{(\nu^2-1)}\,\mathrm{cn}^{-1}(\tfrac{5}{6}\nu^{1/2})\,d\nu, \tag{37}$$

where y is such that

$$\frac{Y}{c_0 t} = \frac{5\sqrt{6}}{12} \frac{(y^2-1)^{1/2}}{y}. \tag{38}$$

The shape of the sonic line is given in figure 4, which shows that for $\delta_2 > 0$ the sonic line is moved downstream. The gradient of the sonic line is everywhere continuous and it reaches its maximum displacement from the plane $X = 0$ when it meets the wall at the point

$$\frac{X}{c_0 t} = -\frac{5\sqrt{3}}{3\pi} \delta_2 \operatorname{cn}^{-1} \tfrac{5}{6}.$$

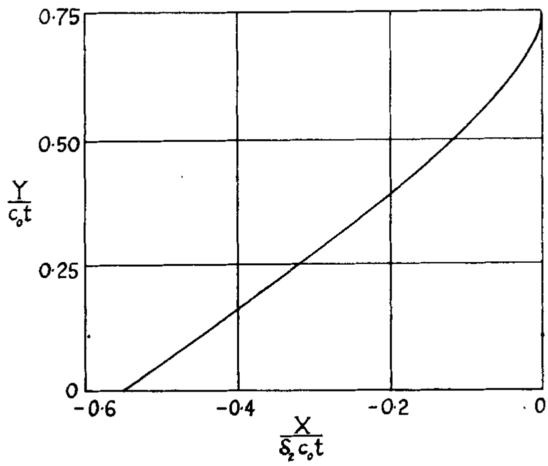

Figure 4. The shape of the sonic line in the perturbed region for the case $\delta_1 = 0$.

6. The nature of the boundaries of the perturbed regions

The above theory predicts singularities in the pressure and velocity fields at the corner and the head and tail of the rarefaction wave. These are just those points in which the boundaries $xy^{1/2} = \frac{5}{6}$, $xy^{1/2} = 1$, meet the wall surface. In the following paragraphs the behaviour at each of the three singular points is discussed briefly and suggestions are made with regard to the behaviour along the remainder of the boundaries.

At the tail of the wave $x = 0$ or $X = -5c_0 t$, the linear theory gives discontinuities in the perturbation velocity of amount

$$\frac{\partial \phi}{\partial X} = -\frac{5\sqrt{6}}{12} c_0(\delta_1 + \delta_2), \qquad \frac{\partial \phi}{\partial Y} = 5c_0(\delta_1 + \delta_2).$$

There is also a singularity in the perturbation pressure where by (33),

$$\frac{p_2 - p_1}{p_1} \sim \frac{35\sqrt{6}c_0}{12c_1} (\delta_1 + \delta_2) \quad \text{as } X \to -5c_0 t. \tag{39}$$

These values satisfy the conditions for a weak shock or rarefaction wave which to the first order lies along the line $2\sqrt{6}Y = X + 5c_0 t$ (both boundaries approximate to this line near the point $X = -5c_0 t$). The gas flows through the shock turning through an angle $\delta_1 + \delta_2$. Hence the pressure change takes the form of a weak shock for $\delta_1+\delta_2 > 0$, and a Prandtl–Meyer expansion for $\delta_1+\delta_2 < 0$.

With the linear theory there is a singularity in the pressure at the corner for $\delta_1 \neq 0$ (see equation (34)). A more accurate picture is obtained from solving the full non-linear equations of motion by a series expansion in terms of the distance from the corner. Thus, it may be shown that for $\delta_1 < 0$ the flow in the neighbourhood of the corner is given by a Prandtl–Meyer fan of the sonic type. The flow upstream of the fan is undisturbed and the pressure change through the fan is $O(\delta_1^{2/3})$. However, for $\delta_1 > 0$ no solution can be obtained by a power series of the above form. The author has shown (Powell 1956, p. 90) that in this case the flow at the corner is just subsonic and disturbances are propagated upstream to a point whose distance from the corner is $O(\delta_1^{2/3})$. The pressure change at the wall surface is continuous and is $O(\delta_1^{2/3})$ but away from the wall it develops into a shock wave.

The circumstances at the head of the wave are different. On the linear theory the pressure behaves as

$$\frac{p_2-p_1}{p_1} \sim \frac{7\sqrt{6}\delta_2}{3\pi}(1-x)^{1/2}, \qquad \text{as } x \to 1, \tag{40}$$

with singularities in the pressure derivatives. In a small region near the head of the wave the full equations of motion may be solved by a series expansion. A solution exists for $\delta_2 < 0$ in which the flow accelerates continuously from rest to a value which is $O(\delta_2^2)$. When $\delta_2 > 0$ disturbances are propagated ahead of the head of the wave $X = c_0 t$. They travel into gas at rest in which the speed of sound is c_0. Hence they take the form of a shock wave which may be shown to have a strength $O(\delta_2^2)$.

We have shown that the extremities of the inner boundary $xy^{1/2} = 1$ are marked by shocks for $\delta_1 > 0$ and expansion regions for $\delta_1 < 0$. This would suggest that the shocks or expansions are continued around the entire boundary although perhaps in a diminished form. This is substantiated by an examination of the normal pressure derivative at the boundary. For a class of problems in cone-field theory, Lighthill (1949 b) has shown that the boundary of a perturbed region is a first approximation to a shock wave when the normal pressure derivative is singular and positive, and to a rarefaction wave otherwise. With certain assumptions, the author has indicated (Powell 1956) how Lighthill's proof may be extended to cover the present case. In this, there is a shock along the inner boundary for $\delta_1 > 0$ and an expansion region for $\delta_1 < 0$ with similar behaviour at the outer boundary for $\delta_2 > 0$ and $\delta_2 < 0$, respectively. However, the strengths of the two shocks differ. The shock at the inner boundary is of strength $O(\delta_1^{4/3})$ increasing to $O(\delta_1^{2/3})$ in the neighbourhood of the corner, whereas the shock at the outer boundary is $O(\delta_2^4)$ increasing to $O(\delta_2^2)$ at the head of the wave.

It is interesting to compare these results with the analogous problem of the diffraction of a shock wave by a corner (Lighthill 1949 a; Tan 1951).

In general, the boundary of the perturbed region behind the incident shock itself represents a shock wave or rarefaction wave whose strength is $O(\delta^2)$. But when the flow behind the incident shock is just sonic, the boundary remains attached to the corner and is locally of strength $O(\delta^{2/3})$ decreasing to $O(\delta^2)$ on the remainder of the boundary. Thus both the present problem and that of Lighthill are similar in the respect that they possess perturbed regions which are bounded by a shock wave or rarefaction wave whose strengths are respectively $O(\delta_1^{2/3})$ and $O(\delta^{2/3})$ at the corner. However, the analysis of the present section suggests that their strengths differ on the remainder of the boundary, being $O(\delta^2)$ in Lighthill's case and $O(\delta_1^{4/3})$ in the present case. This difference can to some extent be explained by the nature of the characteristic lines in the two problems. In Lighthill's problem the general equation of motion is elliptic inside the perturbed region and hyperbolic outside so that in the neighbourhood of the corner only one characteristic is generated which is continued around the entire boundary of the perturbed region. In the present problem, the equation of motion is hyperbolic everywhere (with the exception of a small region near the corner) and all characteristics generated near the corner are continued around the boundary of the perturbed region. It is reasonable to suppose that in the latter case a shock whose strength is of a higher order will be propagated around the boundary.

7. Conclusion

In practice, it is impossible to obtain a perfect vacuum behind the membrane and shock-tube experiments are usually performed with the pressure difference across the membrane obtained by a high compression of the gas on one side. The resulting flow differs from the complete rarefaction wave. When the membrane fractures, a shock wave travels down the tube into the region of rarefaction followed by a region of steady flow. An incomplete rarefaction wave spreads upstream also followed by a region of uniform flow and the two regions of uniform flow are separated by a contact discontinuity which moves downstream with the same velocity as the fluid (e.g. Holder 1953).

Clearly in the above case, the region of perturbation will extend not only through the rarefaction wave but also into the regions of steady flow between the rarefaction wave and the shock wave. However, the solution obtained in the previous sections will still hold in the rarefaction wave since equation (8) is hyperbolic and the flow is determined by conditions on the wall alone.

For the resulting rarefaction wave to be near enough to completeness to extend as far downstream as $(X/c_0 t) = -4{\cdot}5$ requires an initial pressure ratio of thirty million. This is impracticable, but pressure ratios of 10^4 have been obtained and in this case the wave extends downstream to about $(X/c_0 t) = -3$, i.e. two-thirds of the complete rarefaction wave are present.

The author wishes to thank Dr W. Chester for suggesting the subject to him and for much helpful discussion on the problem. During the major part of the research the author was in receipt of a maintenance grant from the Department of Scientific and Industrial Research.

Appendix

(*a*) $\gamma = \frac{5}{3}$

Gases other than air are commonly used in the shock-tube. For example, argon may be used on the low pressure side of the membrane with hydrogen on the high pressure side. The rarefaction wave will always occur in the gas which originally occupies the high pressure region; thus in the above example the previous results will still hold since hydrogen is a diatomic gas.

For monatomic gases such as helium and argon we may approximate to $\gamma = 5/3$. In this case equation (20) may be integrated in terms of elementary functions. Thus the perturbed region is bounded by the curve

$$\left(\frac{X}{c_0 t}+3\right)\left\{1-\frac{1}{k^2}\left(\frac{X}{c_0 t}+3\right)^2\right\}^{1/2} = 2\sqrt{2}\,\frac{Y}{c_0 t},$$

where $k = 4$ for $\delta_2 \neq 0$, $k = 3$ for $\delta_2 = 0$. The boundary extends from $X = c_0 t$ to $X = -3c_0 t$ and apart from a contraction in the X-direction it resembles in general shape the boundary of § 4.

The perturbation velocity along the wall surface is given by

$$\frac{\partial\phi}{\partial X} = \frac{3\sqrt{2}c_0}{2\pi}\left\{\delta_2[2x\cosh^{-1}(1/x)-\cos^{-1}x]+\right.$$
$$\left.+\,\delta_1 H(\tfrac{3}{4}-x)\left[2x\cosh^{-1}\frac{3}{4x}-\cos^{-1}\frac{4x}{3}+\frac{x}{(9-16x^2)^{1/2}}\right]\right\},$$

and the pressure distribution by

$$\frac{p_2-p_1}{p_1} = \frac{5\sqrt{2}}{2\pi}\left\{\delta_2\left[\frac{3}{x}\cos^{-1}x-2\cosh^{-1}\frac{1}{x}\right]+\right.$$
$$\left.+\,\delta_1 H(\tfrac{3}{4}-x)\left[\frac{3}{x}\cos^{-1}\frac{4x}{3}-2\cosh^{-1}\frac{3}{4x}+\frac{1}{(9-16x^2)^{1/2}}\right]\right\}.$$

(*b*) $\gamma = 2$

Results for $\gamma = 2$ are useful for a comparison with hydraulic theory by the so-called hydraulic analogy. In this, the two-dimensional flow of a compressible gas is compared with the flow of water through a channel (e.g. Black & Mediratta 1951).

In this case the boundary of perturbation is given by

$$\left(\frac{X}{c_0 t}+2\right)\left\{1-\frac{1}{k}\left(\frac{X}{c_0 t}+2\right)\right\}^{1/2} = \sqrt{3}\frac{Y}{c_0 t},$$

where $k = 3$ for $\delta_2 \neq 0$, $k = 2$ for $\delta_2 = 0$. The perturbation velocity distribution is

$$\frac{\partial\phi}{\partial X} = \frac{4\sqrt{3}c_0}{3\pi}\left\{\delta_2[2x^{1/2}(1-x)^{1/2}-\cos^{-1}x^{1/2}]+\right.$$
$$\left.+\,\delta_1 H(\tfrac{2}{3}-x)\left[2x^{1/2}(\tfrac{2}{3}-x)^{1/2}-\cos^{-1}\left(\frac{3x}{2}\right)^{1/2}+\frac{(3x)^{1/2}}{6(2-3x)^{1/2}}\right]\right\},$$

and the pressure distribution is given by

$$\frac{p_2-p_1}{p_1} = \frac{8\sqrt{3}}{3\pi x}\left\{\delta_2[2\cos^{-1}x^{1/2} - x^{1/2}(1-x)^{1/2}] + \right.$$
$$\left. + \delta_1 H(\tfrac{2}{3}-x)\left[2\cos^{-1}\left(\frac{3x}{2}\right)^{1/2} - x^{1/2}(\tfrac{2}{3}-x)^{1/2} + \frac{(3x)^{1/2}}{6(2-3x)^{1/2}}\right]\right\}.$$

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